Please indicate how much time you spend on this assignment.

Problem 1
Recall that we use \( \mathbb{N} = \{1, 2, 3, \ldots \} \) to denote the set of natural numbers (Note that \( \mathbb{N} \) does not include 0 to be consistent with the textbook).

For two natural numbers \( a, b \in \mathbb{N} \), we say that \( a \) divides \( b \), and we write \( a \mid b \), if \( b \) can be divided by \( a \) with no remainder. We also say that if \( a \mid b \) that \( b \) is divisible by \( a \). For \( a \in \mathbb{N} \), we also define the set
\[
 a \mid \mathbb{N} = \{ b \in \mathbb{N} \mid a \text{ divides } b \}.
\]
a) Describe in your own words what the set \( a \mid \mathbb{N} \) is.
b) Prove or disprove that \( 12 \mathbb{N} \subseteq 3 \mathbb{N} \).

Problem 2
Let \( \mathbb{N} \) be the universe set for this problem. This means that the only elements considered in this problem are natural numbers. Therefore, if we have a set \( E = \{2, 4, 6, 8, \ldots \} \), the complement of \( E \) is the set \( \overline{E} = \{1, 3, 5, 7, \ldots \} \) of all natural numbers not in \( E \).

Disprove the following statement:
\[
\text{If } A \subseteq \mathbb{N} \text{ and } B \subseteq \mathbb{N}, \text{ then } \overline{A \cup B} = \overline{A} \cup \overline{B}.
\]

Problem 3
The well-ordering principle states that every non-empty subset of \( \mathbb{N} \) has a smallest element. In other words, if \( S \subseteq \mathbb{N} \) and \( S \neq \emptyset \), then there exists an element \( x_{\text{min}} \in S \) such that \( x_{\text{min}} < y \) for all \( y \in S \setminus \{x_{\text{min}}\} \).

Use the well-ordering principle to prove that proofs by induction are valid. More precisely, prove that if \( P : \mathbb{N} \rightarrow \{\text{true, false}\} \) is a predicate that satisfies the two properties
1. \( P(1) = \text{true} \)
2. \( P(n) = \text{true} \implies P(n + 1) = \text{true} \),
then for all \( n \in \mathbb{N} \), \( P(n) = \text{true} \).

You may not use a proof by induction (HINT: use contradiction instead).
Problem 4
Recall that a function $f : A \rightarrow B$ is one-to-one if and only if the following holds: for all $x, y \in A$, if $f(x) = f(y)$, then $x = y$. We also say that $f$ is onto if and only if the following holds: for all $y \in B$, there exists an $x \in A$ such that $f(x) = y$.

Give an example of a function $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ that is onto but is not one-to-one. You do not need to formally prove the correctness of your answer. However, briefly explain why you chose this function.

Bonus Problem: Playing with bits

Sam challenges Charlie and Peter-Michael to a game. Before starting the game, Sam first explains all of the rules and gives Charlie and PM enough time to develop and agree on a strategy.

At the beginning of the game, Charlie and PM are put in separate rooms and cannot communicate directly for the entirety of the game. Once they are separated, the game proceeds in three steps.

1. Sam picks a number $X \in \{1, \ldots, 16\}$ and also chooses a 16-bit string $Y$ (e.g. 0100011010001010). He then shows Charlie both $X$ and $Y$.

2. Charlie then has the option of changing any one bit of $Y$. Let $\hat{Y}$ be the string after Charlie changes it.

3. Sam then shows $\hat{Y}$ to PM, and PM tries to guess what $X$ is.

If PM correctly guesses the number $X$, then Charlie and PM win. However, if PM picks incorrectly, then Sam wins.

a) Prove that Charlie and PM can always win.

b) If the game were modified so that $X \in \{1, 2, \ldots, 2^n\}$ and $Y$ is $2^n$ bits long, prove that Charlie and PM can win for all $n \in \mathbb{N}$.