Please indicate how much time you spent on this assignment.

Problem 1
Let $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \cup \{+, \times, -, \div\}$ be an alphabet.

A string $w \in \Sigma^*$ is called a simple expression if it is in one of the following two forms.

1. $w$ consists of one or more digits with no leading zeros.

2. $w = w_1 \otimes w_2$ where $w_1$ and $w_2$ are also simple expressions and $\otimes \in \{+, \times, -, \div\}$.

Therefore 1074 is a simple expression of the first kind, and $50 + 7$ is a simple expression of the second kind. Similarly, $2 - 700 \times 8 \div 10$ is also a simple expression.

Let $S \subseteq \Sigma^*$ be the set of all such simple expressions.

a) Give a regular expression that recognizes $S$.

b) Give a DFA or an NFA that recognizes $S$.

Problem 2
Let $\Sigma$ be the alphabet from Problem 1, and let $\widehat{\Sigma} = \Sigma \cup \{(, )\}$. ($\widehat{\Sigma}$ contains all the elements of $\Sigma$ along with the left- and right-parentheses.) A string $w \in \widehat{\Sigma}^*$ is called a complex expression if it is in one of the following three forms.

1. $w$ consists of one or more digits with no leading zeros.

2. $w = w_1 \otimes w_2$ where $w_1$ and $w_2$ are also complex expressions and $\otimes \in \{+, \times, -, \div\}$.

3. $w = (w_1)$ where $w_1$ is also complex expression.

Note that all simple expressions are also complex expressions (since the first two forms are identical). However, we now can have expressions with parentheses such as $(2 - 700) \times 8 \div 10$.

Let $C \subseteq \widehat{\Sigma}^*$ be the set of all such complex expressions. Prove that $C$ is not regular using the pumping lemma.
Problem 3
Let $\Sigma$ be an alphabet and let $w \in \Sigma^*$ be a string. We write $w^R$ to denote the reverse of $w$, i.e., if 
$$w = a_1a_2a_3 \cdots a_n$$
where $a_i \in \Sigma$ for each $1 \leq i \leq n$, then 
$$w^R = a_n a_{n-1} \cdots a_1.$$ 
If $A \subseteq \Sigma^*$ is a language, we also define the reversal of $A$ to be 
$$A^R = \{w^R \mid w \in A\}.$$ 
Prove that the class of regular languages is closed under the reverse operator.

Problem 4
Let $E$ be the language of all evenly lengthed bit strings. Therefore, $0111 \in E$, but $000 \notin E$.

a) Prove or disprove that $E \circ E^R$ is regular.

b) Prove or disprove that $\hat{E} = \{ww^R \mid w \in E\}$ is regular.

c) Explain in your own words the difference between the languages $E \circ E^R$ and $\hat{E}$ and why one is regular and the other is not.

(Note that if you give a disproof, you must use the pumping lemma.)

Bonus Problem: Finding the Prefixes

We say a string $x \in \{0, 1\}^*$ is a prefix of a string $y \in \{0, 1\}^*$, and we write $x \sqsubset y$, if $y = xz$ for some string $z \in \{0, 1\}^*$. For a string $x \in \{0, 1\}^*$, we write 
$$P(x) = \{y \in \{0, 1\}^* \mid x \sqsubset y\}$$
to denote the set of all strings for which $x$ is a prefix.

We extend this notation so that for a language $A \subseteq \{0, 1\}^*$, we write 
$$P(A) = \bigcup_{x \in A} P(x).$$
Note that $P(A)$ is simply the language of all the prefixes of the strings in $A$. 

Prove or disprove that if $A$ is a regular language, then $P(A)$ is also regular.