For each problem, please indicate how much time you spent on it.

Problem 1
Prove that the language
\[
\text{DEAD} = \{\langle M, q \rangle \mid M \text{ is a TM and } q \text{ is a state of } M \text{ that is never used}\}
\]
is undecidable. Note that “never used” simply means that \( M \) never enters the state \( q \) on any input.

Problem 2
Let \( \text{FINITE}_{\text{TM}} \) be the language
\[
\text{FINITE}_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is finite}\}.
\]
Prove that \( \text{FINITE}_{\text{TM}} \) is undecidable.

Problem 3
We have seen that \( \text{EQ}_{\text{DFA}} \) is decidable and \( \text{EQ}_{\text{TM}} \) is undecidable. Let \( \text{EQ}_{\text{TD}} \) be the language
\[
\text{EQ}_{\text{TD}} = \{\langle M, D \rangle \mid M \text{ is a TM, } D \text{ is a DFA, and } L(M) = L(D)\}.
\]
Prove that \( \text{EQ}_{\text{TD}} \) is undecidable using a mapping reduction.

Problem 4
Prove or disprove each of the following statements about the relation \( \leq_m \).

a) \( \leq_m \) is reflexive

b) \( \leq_m \) is symmetric

c) \( \leq_m \) is transitive

A proof of the affirmative should include the reduction function and the Turing machine that computes it. A disproof should include a counterexample and an argument why the example causes the statement to fail.
Bonus Problem: How many 1s can we write?

Let $\Gamma = \{0, 1, \sqcup\}$ be the tape alphabet for all TMs in this problem. For $k \in \mathbb{N}$, let $H_k$ be the set of all TMs that have exactly $k$ states and halt on $\epsilon$. For each $\langle M \rangle \in H_k$, let $\#1(M)$ be the number of 1s left on $M$’s tape after running to completion on $\epsilon$.

Define the function $f : \mathbb{N} \to \mathbb{N}$ as

$$f(k) = \max\{\#1(M) \mid \langle M \rangle \in H_k\}.$$ 

In other words, $f(k)$ is the maximum number of 1s any $k$-state TM can possibly write to its tape and not get into an infinite loop.

Prove that $f$ is not a computable function.