Please indicate how much time you spent on this assignment.

Problem 1
Note that the model \((\mathbb{N}, +)\) does not support formulas like \(x = 5\) since the model does not support constants like 5. Prove that for all \(n \in \mathbb{N}\) there is a formula \(\phi_n\) with one free variable \(x\) that has the meaning

\[ \phi_n(x) \equiv [x = n]. \]

Note: In this problem \(\mathbb{N} = \{0, 1, 2, 3, \ldots\}\).

Problem 2
Prove or disprove that \(Th(\mathbb{E}, +, \times)\) is decidable where \(\mathbb{E} = \{0, 2, 4, 6, \ldots\}\) is the set of all non-negative even integers.

Hint: you may use theorems in section 6.2 of the text.

Problem 3
Prove or disprove that \(Th(\mathbb{Z}_5, +, \times)\) is decidable where \(\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}\).

Bonus Problem: Beards Never Grow Back

Titus is considering shaving his beard (gasp!). However, he is convinced that if he does, his beard will never grow back. A proof of this claim is presented below and it relies on the following assumptions.

1. Immediately after you shave your beard, you do not have a beard.

2. Since a beard does not grow in a day, if you do not have a beard on day \(n\) then you do not have a beard on day \(n + 1\).

Proof. (by induction on the number of days since shaving) For the base case, let day 0 be the day Titus shaves his beard. Then assumption 1 tells us that Titus does not have a beard on day 0. For the induction step, assume Titus does not have a beard on day \(n\). Then assumption 2 tells us that he will not have a beard on day \(n + 1\). Thus, we can conclude that Titus will never have a beard ever again.

Does this mean beards never grow back? Explain your answer.