Please indicate how much time you spent on this assignment.

Problem 1
Let \( A, B, C, D \) be languages such that \( A, B \in P \) and let \( C, D \in NP \). For each of the following statements, indicate whether the statement is true or false and justify your answer.

a) \( A \cup B \in P \)

b) \( C \cap D \in NP \)

c) \( \overline{A} \in P \)

Problem 2
Let graphs be undirected in this problem. Define the language

\[
\text{PATH}_\text{SHORT} = \{ (G, x, y, k) \mid G \text{ contains a simple path from } x \text{ to } y \text{ of length at most } k \}.
\]

Prove that \( \text{PATH}_\text{SHORT} \) is in the class \( P \) by giving a polynomial-time Turing machine that decides it.

Problem 3
Let graphs be undirected in this problem. Define the language

\[
\text{PATH}_\text{LONG} = \{ (G, x, y, k) \mid G \text{ contains a simple path from } x \text{ to } y \text{ of length at least } k \}.
\]

Prove that \( \text{PATH}_\text{LONG} \) is \( NP \)-complete.

Problem 4
Let \( U \) be a finite set called a universe set, and let \( S = \{S_1, \ldots, S_n\} \) be a finite collection of sets such that \( S_i \subseteq U \) for each \( 1 \leq i \leq n \). We call \( S \) a family of subsets over universe \( U \).
A sub-family \( \mathcal{C} \subseteq \mathcal{S} \) is \( U \)-total if
\[
U = \bigcup_{S_i \in \mathcal{C}} S_i.
\]
In other words, every element of \( U \) is contained in one of the sets \( S_i \) in the sub-family \( \mathcal{C} \). For example, if \( U = \{1, 2, 3, 4\} \) and \( \mathcal{S} = \{\{1, 2\}, \{3\}, \{4\}, \{3, 4\}\} \), then the set \( \mathcal{C} = \{\{1, 2\}, \{3, 4\}\} \) is \( U \)-total.

Define the set
\[
U\text{-TOTAL} = \{\langle U, \mathcal{S}, k \rangle \mid \mathcal{S} \text{ has a } U\text{-total sub-family of size } k\}.
\]
Prove that \( U\text{-TOTAL} \) is \( NP \)-complete.

**Bonus Problem: 1**

As in problem 4, let \( U \) be a finite universe set, and let \( \mathcal{S} \) be finite family of subsets over \( U \). We say that a set \( H \subseteq U \) is \( \mathcal{S} \)-total if every set \( S_i \in \mathcal{S} \) contains at least one element of \( H \). For example, if \( U = \{1, 2, 3, 4\} \) and \( \mathcal{S} = \{\{1\}, \{1, 2\}, \{3\}, \{3, 4\}\} \), then the set \( H = \{1, 3\} \) is \( \mathcal{S} \)-total.

Define the set
\[
S\text{-TOTAL} = \{\langle U, \mathcal{S}, k \rangle \mid U \text{ has a } \mathcal{S} \text{-total subset of size } k\}.
\]
Prove that \( S\text{-TOTAL} \) is \( NP \)-complete.

**Bonus Problem: 2**

Let \( \{0, 1\} \) be the alphabet for this problem. Define the bounded acceptance problem to be the language
\[
B_{TM} = \{\langle M, w, k \rangle \mid M \text{ is a TM and accepts } w \text{ in at most } k \text{ steps}\}
\]
where \( k \) is a number encoded in binary.

For each of the following statements, indicate whether the statement is true or false and justify your answer.

a) \( B_{TM} \in P \)
b) \( B_{TM} \in NP \)

**HINT:** Think carefully about how the length of \( k \) grows relative to its value.